Test Yourself 1.1

1. V = kg m^2 s^{-3} A^{-1}
2. [G] = kg^{-1} m^3 s^{-2}
3. (a) $\varepsilon_0 = \frac{1}{4\pi}\frac{Q_1Q_2}{Fr^2}$, so $[\varepsilon_0] = \frac{[Q_1][Q_2]}{[F][r^2]} = \frac{\text{C} \times \text{C}}{\text{N} \times \text{m}^2}$ = C^2 N^{-1} m^{-2}
   (b) Using C = A s and N = kg m s^{-2}, $[\varepsilon_0]$ = kg^{-1} m^{-3} s^4 A^2
4. (a) $[h]$ = J s
   (b) $[h]$ = kg m^2 s^{-1}
5. $[\mu_0]$ = H m^{-1} = kg m s^{-2} A^{-2}, so H = kg m^2 s^{-2} A^{-2}
6. F = $[C]$ = kg^{-1} m^{-2} s^4 A^2
7. 2.5 MΩ [= 2.5 × 10^6 Ω]
8. $\left[\frac{1}{\varepsilon_0\mu_0}\right] = \frac{1}{\text{kg}^{-1}\,\text{m}^{-3}\,\text{s}^4\,\text{A}^2 \times \text{kg m s}^{-2}\,\text{A}^2}$ = m^2 s^{-2} = [c^2] QED
9. (a) [σ] = W m^{-2} K^{-4}
   (b) [σ] = M T^{-3} Θ^{-4}
   Note: L cancels out so [σ] does not depend on L
10. $[W]$ = L Θ
11. Ω = $[R]$ = kg m^2 s^{-3} A^{-2}
12. (a) $[c]$ = J kg^{-1} K^{-1}
    (b) $[c]$ = m^2 s^{-2} K^{-1}
13. N s = (kg m s^{-2}) × s = kg m s^{-1}
14. Starting from the rhs and working in dimensions:
    $[p\Delta V] = \left[\frac{F}{A}\right] \times [\Delta V] = \frac{\text{M L T}^{-2}}{\text{L}^2} \times \text{L}^3 = \text{M L}^2\,\text{T}^{-2} = [W]$ QED
15. Working in units
    $[p^2c^2]$ = N^2 s^2 × m^2 s^{-2} = N^2 m^2
    $[m^2c^4]$ = kg^2 m^4 s^{-4} = (kg m s^{-2})^2 m^2 = N^2 m^2
    ∴ The right-hand side is homogeneous.
    $[E^2]$ = J^2 = (N m)^2 = N^2 m^2
    So the two sides have the same units, i.e the equation is homogeneous.
16. Working in dimensions:
    Dimensions of the right side = $[nAve]$ = L^{-3} L^2 (L T^{-1}) (I T) = I = dimensions of the left side QED
17. 6.4 μm s^{-1}
18. If $E_{\text{k max}}$ is expressed in J then the units of both terms on the right must be J, i.e. $[\phi]$ = J.
    If $E_{\text{k max}}$ is expressed in eV then $[\phi]$ = eV.
19. Working in dimensions: $[p] = \left[\frac{F}{A}\right]$ = M L T^{-2} L^{-2} = M L^{-1} T^{-2}
    $[\frac{1}{3}\rho c^2]$ = M L^{-3}(L T^{-1})^2 = M L^{-1} T^{-2}. The two sides have the same dimensions, hence the equation is homogeneous.
20. Working in units: $\left[\frac{h}{\lambda}\right] = \frac{\text{J s}}{\text{m}} = \frac{\text{N m s}}{\text{m}}$ = N s = $[p]$, so the equation is homogeneous.
21. Working in dimensions: $[p]$ = M L^{-1} T^{-2}; [ρ] = M L^{-3};
    ∴ $\left[\sqrt{\frac{\gamma p}{\rho}}\right] = \sqrt{\text{L}^2\,\text{T}^{-2}}$ = L T^{-1}
    $[c]$ = L T^{-1}, so the two sides have the same dimensions, i.e, the equation is homogeneous.
22. Working in units: From Q2, $[G]$ = kg^{-1} m^3 s^{-2}.
    ∴ $\left[-\frac{GM_1M_2}{R}\right] = \frac{\text{kg}^{-1}\,\text{m}^3\,\text{s}^{-2}\,\text{kg kg}}{\text{m}}$ = kg m^2 s^{-2} = $[E]$
    The two sides have the same dimensions, i.e. the equation is homogeneous.
23. $a = \frac{1}{2}$; $b = -\frac{1}{2}$, i.e. $v = c\sqrt{\frac{K}{\rho}}$. Compare this with Q21.
24. $a = b = -\frac{1}{2}$; $c = \frac{3}{2}$, i.e. $T = k\sqrt{\frac{r^3}{GM}}$. Compare this with Kepler's 3rd law.
25. $x = z = \frac{1}{2}$; $y = -\frac{1}{2}$, i.e. $c = k\sqrt{\frac{Tl}{m}}$. In fact it is usually written $c = \sqrt{\frac{T}{\mu}}$, where μ is the mass per unit length of the wire. The dimensionless constant k = 1.

Test Yourself 2.1

1	23	2	−11	3	16	4	52	5	306
6	21 000	7	600	8	42	9	520	10	264
11	75	12	40	13	−3	14	5	15	3.33
16	6	17	0.20	18	−0.5	19	±12	20	±6
21	2	22	−2.1726	23	±44.3	24	1.25	25	8

Test Yourself 2.2

1. $m = \frac{E}{c^2}$
2. $R = \frac{V^2}{P}$
3. $\rho = \frac{RA}{l}$
4. $f = \frac{c}{\lambda}$
5. $r = \sqrt{\frac{I}{4\pi\sigma T^4}}$ or $\frac{1}{T^2}\sqrt{\frac{I}{4\pi\sigma}}$ etc.
6. $c = \sqrt{\frac{3p}{\rho}}$
7. $t = \frac{v-u}{a}$
8. $u = \sqrt{v^2 - 2as}$
9. $t = \frac{2s}{u+v}$
10. $v = \frac{I}{nAe}$
11. $x = \sqrt{\frac{2E}{k}}$
12. $g = \frac{4\pi^2 l}{T^2}$
13. $v = \sqrt{2gh}$
14. $m = \frac{Ft}{v-u}$
15. $v = \frac{s - \frac{1}{2}at^2}{t}$ or $\frac{s}{t} - \frac{1}{2}at$
16. $h = \frac{E_{\text{k max}} + \phi}{f}$
17. $M_2 = \frac{Fr^2}{GM_1}$
18. $M = \frac{4\pi^2 a^3}{GT^2}$

19 $X = \sqrt{Z^2 - R^2}$

20 $\beta = \sqrt{1 - \frac{m_0^2}{m^2}}$

21 $C = \frac{C_1C_2}{C_1 + C_2}$ or $\frac{1}{\frac{1}{C_1} + \frac{1}{C_2}}$

22 $r = \frac{ER}{V} - R$ or $R\left(\frac{E}{V} - 1\right)$

23 $M_1 = \frac{M_2 d}{r_1} - M_2$ or $M_2\left(\frac{d}{r_1} - 1\right)$

24 $M_2 = \frac{M_1}{\left(\frac{d}{r_1} - 1\right)}$ or $\frac{r_1M_1}{d - r_1}$

25 $M_1 = \frac{4\pi^2 d^3}{T^2G} - M_2$

Test Yourself 2.3

1 $3x + 6$
2 $20x + 24$
3 $a - 3$
4 $20 + 10a + 15b$
5 $xy - 2x + 3y - 6$
6 $x^2 - 4y^2$
7 $x^2 + 10x + 25$
8 $4 - 4y + y^2$
9 $2p^2 + pq - 3q^2$
10 $25a^2 - 60ab + 36b^2$
11 $-9 + 6x - x^2$
12 $ax - ab$
13 $x^2 - a^2$
14 $x^2 - 4ax + 4a^2$
15 $z^2 + b^2$
16 $z^2 + b^2$
17 $4zb$
18 $t^4 + 2t^2 + 1$
19 $t^4 - 1$
20 $t^3 - 2t^2 + t - 2$
21 $a^3 + a^2b - ab^2 - b^3$
22 $a - b$
23 $a + b$
24 1
25 $x - c$

Test Yourself 2.4

1 5.4
2 12.5
3 960
4 4.44
5 10
6 26.7
7 3.33
8 6.66
9 30
10 487
11 12
12 5.97×10^{24}
13 1.77×10^{-3}
14 1.96×10^{-5}
15 2.19
16 1245
17 314
18 1.89×10^{-7}
19 9.95×10^{26}
20 25.9
21 2.5
22 1.05
23 20
24 −24
25 1.98×10^{8}

Test Yourself 3.1

1 $x = \pm 4$
2 $x = \pm 0.2$
3 $t = 0$ or 7
4 $t = 0$ or 30
5 $t = 0$ or 10.2
6 $v = \pm 77.5$
7 $v = \pm 3460$
8 $x = \pm 7$
9 $l - 0.24 = \pm 5.57$, $\therefore l = -5.13$ or 5.61
10 $v + 50 = \pm 70.7$, $\therefore v = -120.7$ or 20.7
11 $v - 5 = \pm 25.2$, $\therefore v = -20.2$ or 30.2
12 $x = 1$ or -2
13 $x = -2.55$ or -0.79
14 $t = 0.76$ or 13.24
15 $t = 0.43$ or 11.8
16 $t = 6.95$ or 18.05
17 $x = \pm 2$ m. NB. units!
18 $v = \pm 1000$ m s^{-1}
19 $t = 2.04$ s. NB. The 0 solution is incorrect as the question asked for the time at which the stone **returned** to the ground.
20 57 km s^{-1}.
21 $t = 1.36$ s [ignore the negative root].
22 3500 m, ignoring the 0 root.
23 Total distance from centre = 11 530 km; $h = 5150$ km.
24 20 m s^{-1}.
25 2.70 s.

Test Yourself 3.2

1 $a = 3.5$; $u = 10$
2 $r = 0.5$; $E = 2.0$
3 $a = 1.5$; $u = 4.0$
4 $r = 3.0$; $E = 2.25$
5 $a = 4$; $v = 24$
6 $v = 15$; $m = 10$
7 $k = 25$; $l_0 = 0.2$
8 $u = \pm 6$; $a = 2$
9 $a = 0.75$ m s^{-2}; $u = 2.5$ m s^{-1}. [NB. units]
10 $a = 0.45$ m s^{-2}; $u = \pm 6.78$ m s^{-1}.
11 $r = 1.5\ \Omega$; $E = 6.0$ V
12 $u = 8$ m s^{-1}; $a = 3$ m s^{-2}.
13 (a) $I_1 = 0.0978$ A; $I_2 = 0.0434$ A
(b) $V_{2V} = 1.90$ V; $V_{1.5V} = 1.41$ V
(c) $V_{10}\ \Omega = 1.41$ V = the pd across the 1.5 V cell as expected.
14 $E = 12$ V; $r = 12\ \Omega$.
15 Solution 1: $v_1 = 5$ m s^{-1}; $v_2 = 8$ m s^{-1}. Solution 2: $v_1 = 7$ m s^{-1}; $v_2 = 4$ m s^{-1}
16 $v_1 = -\frac{4}{3}$ m s^{-1}; $v_2 = \frac{8}{3}$ m s^{-1}. The other solution with $v_1 = 4$ m s^{-1} and $v_2 = 0$ represents a near miss!
17 $R = 6.85\ \Omega$; $\varepsilon = -0.023$ V
18 $R = 4.80\ \Omega$; $\varepsilon = 0.013$ A
19 $\mu = 0.053$ kg; $k = 25.2$ N m^{-1}
20 $h = 2.531$ m; $g = 9.82$ m s^{-2}.
21 $u = 10$ m s^{-1}; $a = 2.0$ m s^{-2}.
22 Solution 1: $u = 15$ m s^{-1}; $a = 5$ m s^{-2} (constant acceleration). Solution 2: $u = 25$ m s^{-1}; $a = 0$ (constant velocity).

23 The valid solution is $r = 2.0\ \Omega$, $E = 24$ V. The invalid solution has $r = -14\ \Omega$!

24 (a) $T_1^2 = 4\pi^2 \frac{M_1}{k}$. With the additional mass, $T_2^2 = 4\pi^2 \frac{M_1 + M_2}{k}$

Subtracting gives $T_2^2 - T_1^2 = 4\pi^2 \frac{M_2}{k}$ as required.

(b) $k = 18.0\ \text{N m}^{-1}$; $M_1 = 0.200$ kg.

25 As in Q24, $T_2^2 - T_1^2 = 4\pi^2 \frac{\Delta l}{g}$, where Δl is the change in length $= -0.500$ m.

$g = 8.657\ \text{m s}^{-2}$; original length $= 2.500$ m

Test Yourself 3.3

1 $\sqrt{(1+x)^3} = 1 + \frac{3}{2}x + \frac{\frac{3}{2} \times \frac{1}{2}}{2 \times 1}x^2 + \frac{\frac{3}{2} \times \frac{1}{2} \times \left(-\frac{1}{2}\right)}{3 \times 2 \times 1}x^3 + \frac{\frac{3}{2} \times \frac{1}{2} \times \left(-\frac{1}{2}\right) \times \left(-\frac{3}{2}\right)}{4 \times 3 \times 2 \times 1}x^4 + \ldots$

$= 1 + \frac{3}{2}x + \frac{3}{8}x^2 - \frac{1}{16}x^3 + \frac{3}{128}x^4 + \ldots$

2 1.837117

3 1; 0.75; 0.09375; −0.0078125; 0.0014648

4 1; 1.75; 1.84375; 1.83594; 1.83740 (a) 4.7%, (b) −0.4%

5 Calculator value = 1.15369

Terms: 1; 0.15; 0.00375; −0.0000625; 0.000000234

Totals: 1; 1.15; 1.15375; 1.15369; 1.15369

(a) 0.3%, (b) 4×10^{-3}%

6 1.03

7 1.03

8 0.94

9 1.15

10 1.1

11 $\sqrt{4.5} = \sqrt{4 \times (1 + 0.125)} = 2 \times (1 + 0.125)^{0.5}$

$\therefore \sqrt{4.5} = 2 \times \left(1 + 0.5 \times 0.125 + \frac{0.5 \times (-0.5)}{2 \times 1} \times 0.125^2 + \ldots\right)$

1st order approximation = 2.125

2nd order approximation = 2.121…

12 $\sqrt[3]{1100} = 10 \times \sqrt[3]{1 + 0.1} = 10 \times (1 + 0.033\ldots) = 10.33\ldots$ to 1st order.

13 $\frac{1}{\sqrt{1+x}} = (1+x)^{-0.5}$

$= 1 - 0.5x + \frac{-(0.5) \times (-1.5)}{2 \times 1}x^2 + \frac{-(0.5) \times (-1.5) \times (-2.5)}{3 \times 2 \times 1}x^3 + \frac{-(0.5) \times (-1.5) \times (-2.5) \times (-3.5)}{4 \times 3 \times 2 \times 1}x^4 + \ldots$

$= 1 - 0.5x + 0.375x^2 - 0.3125x^3 + 0.27344x^4 + \ldots$

14 Terms to 4th order: 1; 0.1; 0.015; 0.0025; 0.00044

Partial sums: 1; 1.1; 1.115; 1.1175; 1.11795

Calculator value = 1.11803

15 To 1st order: $(1+x)^n - (1-x)^n = (1 + nx\ldots) - (1 - nx)$

$= 1 + nx - 1 + nx = 2nx$

16 To 1st order: $\sqrt{1+x} - \sqrt{1-x} = 1 + \frac{1}{2}x - \left(1 - \frac{1}{2}x\right) = x$.

17 To 1st order: $(1+x)^n - \frac{1}{(1+x)^n} = (1 + nx) - (1 - nx) = 2nx$

18 To 1st order: $(x+a)^n = x^n\left(1 + \frac{a}{x}\right)^n = x^n\left(1 + \frac{na}{x}\right) = x^n + nax^{n-1}$

This will be a good approximation if $na << 1$

19 To 1st order: $(x+a)^n - x^n = nax^{n-1}$

20 (a) $AC = \sqrt{1.000^2 + 0.020^2} = (1 + 0.0004^2)$

$= 1 + \frac{1}{2} \times 0.0004 - 1.0002$ to 1st order.

(b) AC = 1.00019998

21 (a) $S_1P = \sqrt{1^2 + 0.00225^2} = (1 + 5.0625 \times 10^{-6})^{0.5}$

$= 1 + 2.53 \times 10^{-6}$ m

$S_2P = \sqrt{1^2 + 0.00175^2} = (1 + 3.0625 \times 10^{-6})^{0.5}$

$= 1 + 1.53 \times 10^{-6}$ m

$\therefore S_1P - S_2P = 1.00 \times 10^{-6}$ m.

(b) 1.00×10^{-6} m

22 $S_1P = \sqrt{D^2 + \left(x + \frac{d}{2}\right)^2} = D\left(1 + \frac{\left(x + \frac{d}{2}\right)^2}{D^2}\right)^{\frac{1}{2}} = D\left(1 + \frac{\left(x + \frac{d}{2}\right)^2}{2D^2}\right)$ to 1st order.

$S_2P = \sqrt{D^2 + \left(x - \frac{d}{2}\right)^2} = D\left(1 + \frac{\left(x - \frac{d}{2}\right)^2}{D^2}\right)^{\frac{1}{2}} = D\left(1 + \frac{\left(x - \frac{d}{2}\right)^2}{2D^2}\right)$ to 1st order.

$\therefore S_1P - S_2P = \frac{\left(x + \frac{d}{2}\right)^2}{2D} - \frac{\left(x - \frac{d}{2}\right)^2}{2D} = \frac{x^2 + xd + \frac{d^2}{4} - \left(x^2 - xd + \frac{d^2}{4}\right)}{2D} = \frac{xd}{D}$

This leads on to the Young Fringes formula.

23 To 2nd order:

$\sqrt{1+x} + \frac{1}{\sqrt{1+x}} = \left(1 + \frac{1}{2}x + \frac{\frac{1}{2}\left(-\frac{1}{2}\right)}{2 \times 1}x^2\right) + \left(1 - \frac{1}{2}x + \frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{2 \times 1}x^2\right)$

$= 1 + \frac{1}{2}x - \frac{1}{8}x^2 + 1 - \frac{1}{2}x + \frac{3}{8}x^2$

$= 2 + \frac{1}{4}x^2$

24 To 2nd order:

$(1+x)^n + (1+x)^{-n} = 1 + nx + \frac{n(n-1)}{2}x^2 + \left(1 - nx + \frac{n(n-1)}{2}x^2\right)$

$= 2 + n^2x^2$

With $n = 4$ and $x = 0.1$ this gives 2.16. The calculator value is 2.15

25 $\sin\theta = \frac{\text{opposite}}{\text{hypotenuse}} = \frac{x}{\sqrt{1+x^2}} = x\left(1 - \frac{1}{2}x^2\right)$ x to 3rd order.

$\cos\theta = \frac{\text{adjacent}}{\text{hypotenuse}} = \frac{1}{\sqrt{1+x^2}} = 1 - \frac{1}{2}x^2$ to 3rd order.

$\tan\theta = \frac{\text{opposite}}{\text{adjacent}} = x$ exactly!

Test Yourself 4.1

1 (a) 5 (b) 25 (c) 0.2 or $\frac{1}{5}$

(d) 0.04 or $\frac{1}{25}$ (e) 625

2 (a) 4 (b) $0.25/\frac{1}{4}$ (c) 8

(d) 128 (e) $0.125/\frac{1}{8}$

3 (a) $a^{\frac{1}{4}}/a^{0.25}$ (b) $a^{-\frac{1}{4}}/a^{-0.25}$ (c) $a^{\frac{2}{3}}/a^{0.667}$
(d) $a^{\frac{2}{5}}/a^{0.4}$ (e) $a^{-\frac{3}{2}}/a^{-1.5}$

4 (a) 15 (b) 15 (c) 0.16 (d) 2.5

5 $p = -2$

6 $p = \frac{3}{2}$

7 $p = \frac{3}{2}$, $k = \dfrac{1}{6\sqrt{\pi}}$

8 $R = \dfrac{16\rho V}{\pi^2 d^4}$, i.e. $k = \dfrac{16\rho V}{\pi^2}$ and $n = -4$

9 (a) $2000 \times L_{\odot} = 8 \times 10^{29}$ W
(b) $0.0081 \times L_{\odot} = 3 \times 10^{24}$ W

10 $R = 5I^{-\frac{2}{3}}$, i.e. $c = 5$ and $n = -\frac{2}{3}$

11 (a) 0.6020 (b) −1.3980 (c) 0.9030
(d) 2.3010 (e) 0.3980
[Part (e) $\log 2.5 = \log\dfrac{10}{4} = \log 10 - \log 4 = 1.0000 - 0.6020$]

12 (a) 3.170 (b) −1.585 (c) 2.585
(d) 0.585 (e) 1.262
$\left[\text{Part (e) } \log_3 4 = 2\log_3 2 = \dfrac{2}{\log_2 3}\right]$

13 (a) 2.0 (b) −1.0 (c) $0.5 / \frac{1}{2}$
(d) $1.5 / \frac{3}{2}$ (e) $-1.25 / -\frac{5}{4}$

14 (a) $0.5 / \frac{1}{2}$ (b) $2.5 / \frac{5}{2}$ (c) −3
(d) $0.25 / \frac{1}{4}$ (e) 2.16
$\left[\text{Part (e) } \log_4 20 = \log_4 2 + \log_4 10 = 0.5 + \dfrac{1}{\log_{10} 4} = 0.5 + \dfrac{1}{2\log 2}\right]$

15 (a) $5\log 2$ (b) $-\log 2$ (c) 0 (d) $-\log 2$

16 (a) $2\ln 2 + 1$ (b) $3\ln 2 + 1$ (c) $5\ln 2 - 1$
(d) $4\ln 2 - 1$ (e) $\frac{1}{2}\ln 2 - 2$

17 (a) $x = 0.90$ (b) $x = -0.90$ (c) $x = 4.61$
(d) $x = 7.97$ (e) $x = 403$

18 (a) Remember that $e^{\ln b} = b$
$x\ln a = \ln a^x \therefore e^{x\ln a} = e^{\ln a^x} = a^x$ QED
(b) $2^{\pi} = e^{\pi \ln 2} = e^{3.142 \times 0.6931} = 8.82$

19 (a) $x = 16$ (b) $x = \pm 8$ (c) $x = 6.87 \times 10^{10}$
(d) $x = \pm\frac{1}{2}$ (e) $x = 36$

20 (a) $L_1 = 10\log\dfrac{1}{10^{-12}} = 10\log 10^{12} = 10 \times 12 = 120$ dB SIL
(b) $L_1 = 10\log(10^{12}I)$ (1)
Consider an increase of 3 dB; let the sound intensity be kI
Then $L_1 + 3 = 10\log(10^{12}kI)$
$\therefore L_1 + 3 = 10\log k + 10\log(10^{12}I)$
Subtract equation (1). $\therefore 3 = 10\log k$.
$\therefore \log k = 0.3$, $\therefore k = 2.00$ [3 s.f.]

21 (a) 1.980×10^6 s (b) 96 Bq (c) 11.7×10^6 s.

22 (a) $f_{35} = 1.55$ Hz; $f_{45} = 1.06$ Hz
(b) Substituting the values of l and f into $f = kl^n$:
$1.55 = k \times 0.35^n$ (1) and $1.06 = k \times 0.45^n$ (2)
Dividing equation (1) by equation (2) → $1.462 = 0.778^n$
Taking natural logs: → $\ln 1.462 = n\ln 0.778$
→ $n = -1.51$ [$\log_{10}$ can be used here instead]
Substituting into equation (1) → $k = \dfrac{1.55}{0.35^{-1.51}} = 0.32$
Alternative method: take logs of equations (1) and (2) and solve the resulting simultaneous equations for k and n.
(c) Plot a graph of $\ln f$ against $\ln l$ [or $\log f$ against $\log l$]. The graph should be a straight line with a negative gradient. The value of n is the gradient. The intercept on the $\log f$ axis is the value of $\log k$, so $k = 10^{\text{intercept}}$.

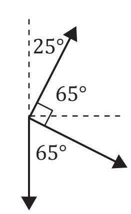
23 (a) Graph of $\ln C$ against x should be plotted [units of C and x can remain in min^{-1} and cm].
The gradient of the graph should be ~ −0.49 and the intercept on the $\ln C$ axis ~6.3.
$\therefore \dfrac{1}{L} = 0.49$ giving a value of $L = 2.04$ cm
$\ln C_0 = 6.3 \therefore C_0 = 540\ \text{min}^{-1}$
(b) $25 = 540e^{-\frac{x}{2.04}}$. $\therefore -\dfrac{x}{2.04} = \ln\left(\dfrac{25}{540}\right) \rightarrow x = 6.3$ cm.
[i.e. an additional shielding of 5.8 cm]

24 (a) A graph of $\ln I$ against $\ln V$ has a gradient of ~0.547 and intercept of ~−0.729 on the $\ln I$ axis. These give $n = 0.55$ [2 s.f.] and $k = 0.48$ [2 s.f.]
(b) $c = k^{-1} = 2.08$. $m = 1 - n = 0.45$

25 (a) $n = \dfrac{60}{8} = 7.5$. $\therefore A = 800 \times 2^{-7.5} = 4.42$ kBq
(b) $\lambda = \dfrac{\ln 2}{8} = 0.0866\ \text{day}^{-1}$.
$\therefore A = 800e^{-0.0866 \times 100} = 0.138$ kBq $= 138$ Bq.
(c) (i) Gradient = $-\ln 2$; intercept = $\ln A_0$
= 6.68 [with A in kBq]
(ii) Gradient = $-\lambda = 0.0866\ \text{day}^{-1}$;
intercept = $\ln A_0$ i.e. same as in (i).

Test Yourself 5.1

1



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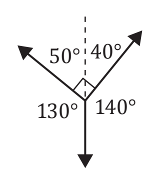

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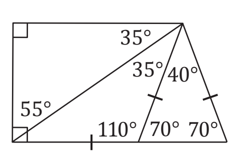

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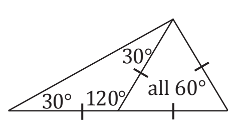

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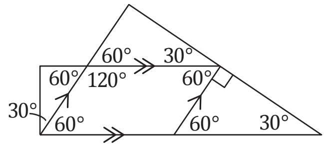


6 (a) 173 mm (b) 100 mm

7 (a) 47.7 m (b) 62.2 m

8 (a) 35.8 cm (b) 46.7 cm

9 (a) 180 mm (b) 56.3°

10 (a) 48.2° (b) 22.4 m

11 $x = 150$ m; $y = 260$ m

12 (a) height = 140 m (b) distance = 300 m

13 1015 m

14 (a) 34.8° (b) 49.3° (c) 61.0°

15 (a) 35.2°

(b) $n_2 = 1.52$ is irrelevant. ϕ would be the same even if this layer were not there.

16 (a) $n = 1.60$ (b) 40.6°

17 $n = 1.58$

18 $n = 1.39$

19 $n = 1.40$

20 $n = 1.41$ [θ must be 45° and angle of incidence must be 90°]

21 (a) $\cos\alpha = \sqrt{1-\sin^2\alpha} = \sqrt{1-0.8^2} = \pm 0.6$

(b) $\tan\alpha = \dfrac{\sin\alpha}{\cos\alpha} = \dfrac{0.8}{\pm 0.6} = \pm 1.33$

22 $\cos 2\beta = \cos(\beta+\beta) = \cos\beta\cos\beta - \sin\beta\sin\beta = \cos^2\beta - \sin^2\beta$

But $\sin^2\beta = 1 - \cos^2\beta$

$\therefore \cos 2\beta = \cos^2\beta - (1-\cos^2\beta) = 2\cos^2\beta - 1$ QED

23 (a) $\cos\chi = \sqrt{1-\sin^2\chi} = \pm\sqrt{1-x^2}$

(b) $\cos(180° + \chi) = \cos 180° \cos\chi - \sin 180° \sin\chi$

$= -1 \times \cos\chi - 0 \times \sin\chi$

$\therefore \cos(180° + \chi) = -\cos\chi = \pm\sqrt{1-x^2}$

(c) $\tan(360° - \chi) = \dfrac{\sin(360° - \chi)}{\cos(360° - \chi)}$

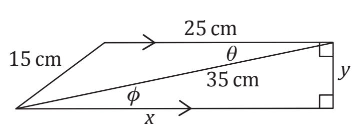
$= \dfrac{\sin 360° \cos\chi - \cos 360° \sin\chi}{\cos 360° \cos\chi + \sin 360° \sin\chi}$

$\cos 360° = \cos 0° = 1$ and $\sin 360° = \sin 0° = 1$

$\therefore \tan(360° - \chi) = \dfrac{-\sin\chi}{\cos\chi} = \dfrac{-x}{\pm\sqrt{1-x^2}}$

$= \pm\dfrac{x}{\sqrt{1-x^2}}$

24 Applying the cosine rule:



$15^2 = 25^2 + 35^2 - 2 \times 25 \times 35 \cos\theta$

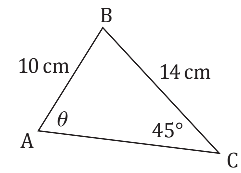
$\therefore \theta = 21.8°$

$\therefore \phi = 21.8°$ [alternate angles]

$\therefore y = 35 \sin 21.8° = 13.0$ cm

and $x = 35 \cos 21.8° = 32.5$ cm.

25 First draw the triangle [not to scale].



Applying the sine rule: $\dfrac{14}{\sin\theta} = \dfrac{10}{\sin 45°}$

$\therefore \theta = \sin^{-1}\left(\dfrac{14 \times \sin 45°}{10}\right) = 81.87°$ or $98.13°$

$\therefore \hat{B} = 180° - (\theta + 45°) = 53.13°$ or $36.87°$

$AC^2 = 10^2 + 14^2 - 2 \times 10 \times 14 \cos B \therefore AC = 11.3$ cm or 8.5 cm

Alternatively: apply the cosine rule directly: Put AC = x

$10^2 = 14^2 + x^2 - 2 \times 14x \cos 45°$

Solve this quadratic equation for x.

26 $\theta = \sin^{-1} 0.5 = \frac{1}{6}\pi$ or $\frac{5}{6}\pi$ or

$\theta = \sin^{-1}(-0.25) = -0.253$ or -2.889

27 (a) $\theta = \sin^{-1}\left(\pm\dfrac{2}{\sqrt{5}}\right) = 1.107$ or -2.034.

Note: −1.107 and +2.034 are not solutions. The process of squaring introduces spurious solutions

(b) $\theta = 0.262$ rad or $0.262 - \pi$ rad $= -2.880$ rad.

(c) $\theta = \pm 0.524$ rad

(d) $\theta = 0$ or ± 2.094 rad

(e) $\theta = \pm 1.57$ rad or 0.252 rad

28 $\sqrt{2}\sin\alpha + \sqrt{2}\cos\alpha$

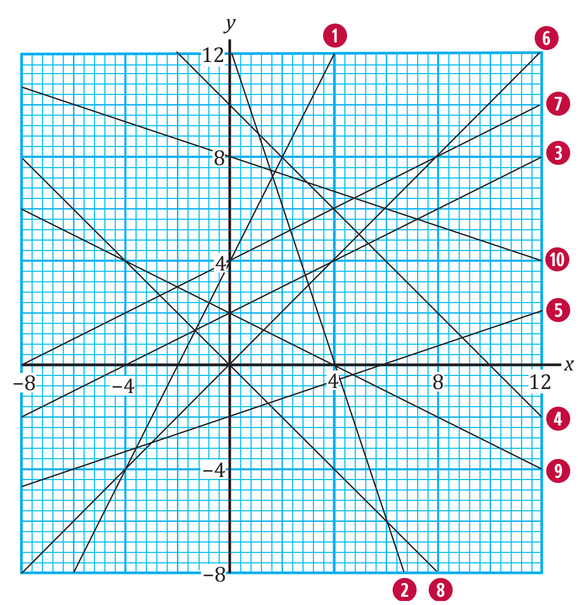
29 (a) $25 \sin(\alpha - 1.287 \text{ rad})$

(b) $25 \cos(\alpha - 2.858 \text{ rad})$

30 $\phi = -0.643$ rad.

Test Yourself 6.1

Questions 1–10



11 $y = 1.5x - 6$

12 $y = -0.4x + 30$

13. $V = -0.2I + 6.0$
14. $V = 4.14 \times 10^{-15}f - 0.70$
15. $v = 0.8t + 16$
16. $F = 25l - 5.0$
17. $V = -1.33I + 3.07$
18. $v = -0.2t + 26$
19. $F = 0.5l - 3$
20. $V = 5 \times 10^{-15}f - 1.5$
21. $V = 9.6 - 4.0I$, $E = 9.6$ V; $r = 4.0\ \Omega$
22. $k = 1.06$ N cm^{-1}, $l_0 = 4.42$ cm
23. $a = 8$ m s^{-2}; $u = 11600$ m s^{-1} [or 0.008 km s^{-2} and 11.6 km s^{-1}]
24. [Gradient = 4.2×10^{-15} [V s], intercept = −0.60 [V]], leading to $h = 6.7 \times 10^{-34}$ J s and $\phi = 9.6 \times 10^{-20}$ J [= 0.6 eV]

Test Yourself 6.2

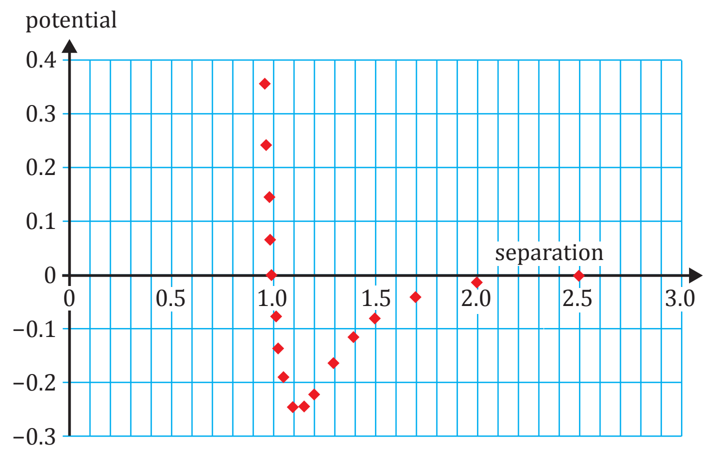
The solutions given are the least squares fit solutions. For graphs drawn freehand, slightly different, but equally acceptable, answers will be obtained.

1. Gradient −1.0 [Ω]; intercept 6.12 [V]. So emf = 6.12 V; internal resistance = 1.0 Ω
2. Gradient 0.21 [m s^{-2}], intercept 3.63 [m s^{-1}]. So initial velocity = 3.63 m s^{-1}; acceleration = 0.21 m s^{-2}.
3. Gradient 0.228 [N cm^{-1}]; intercept −1.13 [N]. So spring constant = 0.228 N cm^{-1}, unloaded length = 4.9 cm.
4. Gradient 0.0036 [atm °C^{-1}], intercept 0.945 [atm]; So p_0 = 0.945 atm and absolute zero (from data) = −263 °C.
5. Gradient −0.0469 [V mA^{-1}]; intercept 10.5 [V]; Emf = 10.5 V; internal resistance = 47 Ω.
6. The graph of $\sqrt{s}$ against t is a straight line with a gradient 1.14 and an intercept of 0.073 on the $\sqrt{s}$ axis [LSF]. This is close enough to a zero intercept to verify the relationship. The acceleration is 2× gradient^2 = 2.6 m s^{-2}.
7. Graph v^2 against s. It is straight with gradient 0.562 and intercept 404. The acceleration $a = \frac{1}{2} \times$ gradient = 0.28 m s^{-2}. The intercept is u^2 so u = 20 m s^{-1}.
8. Plot f against $1/l$ on a restricted axis [e.g. 240 – 520 Hz and 2.4 – 5.0 m^{-1}]. Other possibilities are $1/f$ against l or the axis may be the other way around. Using f v $1/l$ the intercept on the f axis is 1.2 Hz [LSF] which is close to zero and hence consistent with the relationship. The gradient is $c/2$ = 104 [m s^{-1}], so c = 208 m s^{-1}.
9. As in 6.5.2 the graph should be l against $1/f$. The gradient is $c/4$ and the intercept $-\varepsilon$. The graph is straight with gradient 8580 [cm s^{-1}] and intercept −1.3 [cm] giving the speed of sound as 34320 cm s^{-1} [342 m s^{-1}] and end correction 1.3 cm.
10. A graph of T^2 against l should be straight with a gradient of $4\pi^2/g$ and intercept $4\pi^2\varepsilon/g$. The graph has a gradient of 4.11 [s^2 m^{-1}] and intercept 0.082 [s^2]. This gives g = 9.6 m s^{-2} and ε = 2 cm.
11. A graph of d against $1/\sqrt{R}$ should be straight with gradient $\sqrt{k}$ and intercept $-\varepsilon$. The graph has a gradient of 236 and an intercept on the d axis of −1.8. This gives a value for k as 56 000 cpm cm^2, and ε as 1.8 cm.
12. A graph of T^2 against l^2 should be straight with gradient $\frac{2m}{k}$ and intercept $\frac{I}{k}$ on the T^2 axis. The graph is straight with gradient 5600 [s^2 m^{-2}] and intercept 28.5 [s^2]. With m = 0.1 kg this gives a value of k of 3.6×10^{-5} kg m^2 s^{-2} [or, N m rad^{-1}] and $I = 1.0 \times 10^{-3}$ kg m^2.
13. A graph of T^2y against y^2 should be a straight line with gradient $\frac{4\pi^2}{g}$ and intercept $\frac{4\pi^2k^2}{g}$ on the T^2y axis. The graph is a straight line of gradient 4.00 [s^2 m^{-1}] and intercept 0.76 [s^2 m] on the T^2y axis. This gives g = 9.87 kg m^{-2} [or N kg^{-1}] and k = 0.43 m.
14. A graph of $\frac{1}{V}$ against $\frac{1}{R}$ should be straight with gradient $\frac{r}{E}$ and intercept $\frac{1}{E}$ on the $\frac{1}{V}$ axis. The graph is straight with a gradient of 0.219 [Ω V^{-1}] and intercept 0.103 [V^{-1}]. This gives a values of E as 9.7 V and r as 2.1 Ω.
15. A graph of $\frac{1}{v}$ against $\frac{1}{u}$ [or vice versa] should be a straight line of gradient −1 with an intercept on either axis of $\frac{1}{f}$. The graph has a gradient of −1.00 as predicted and an intercept of 0.0679 on the $\frac{1}{v}$ axis, giving a value for f of 14.7 cm.
16. The graph of $\sin\theta_2$ against $\sin\theta_1$ is straight with a gradient of 0.803 and an intercept of 0.0014 on the $\sin\theta_2$ axis, which is consistent with passing through the origin. Hence $\sin\theta_1 \propto \sin\theta_1$. The speed of light in glass is 0.803 × the speed in water.

 Speed of light in water = $\frac{3.00}{1.33} \times 10^8$ m s^{-1}. This gives the speed of light in glass as 1.81×10^8 m s^{-1}.

Data Exercise 6.1

E_p minimum = −0.245, at a separation of 1.12–1.13



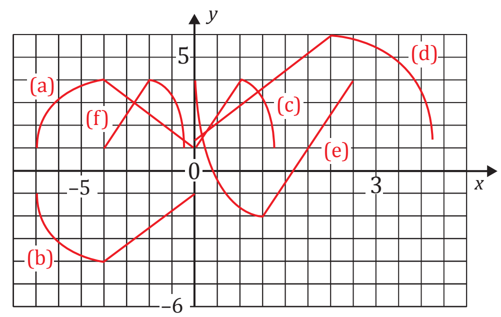
E_p minimum = −0.245, at a separation of 1.12–1.13

Data Exercise 6.2

The LSF graph has a gradient of 1.31 [m s^{-2}] and intercept of 0.006 [m] on the s axis. This is consistent with a constant acceleration of 2.6 m s^{-2} and initial value of $s = 0$.

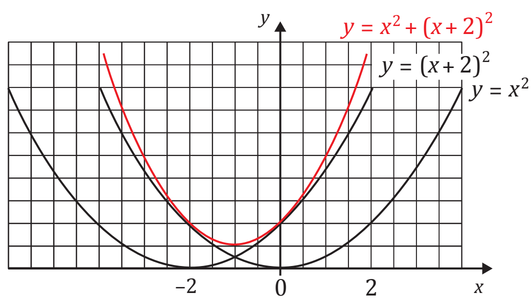
Test Yourself 6.3

1



2 f^{-1} is only defined between $x = 0$ and $x = 7$.

3



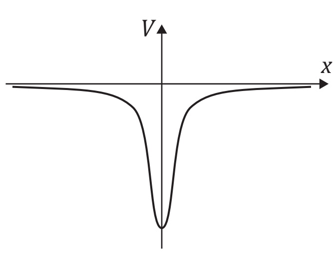
4 (a) The function is symmetrical about $x = 0$ and is >0 for all values of x. As $x \to \pm\infty$, $g \to +\infty$. The magnitude of the gradient increases as $|x|$ increases.

(b) Minimum at (0, 0.2)

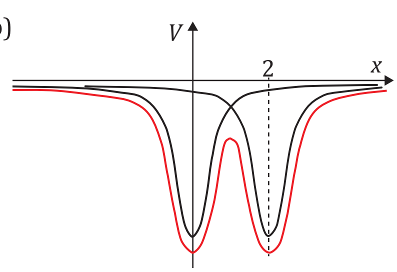
5 (a) (i) Minimum at (0, −5)

(ii) Points of inflexion at $\left(\pm\frac{1}{\sqrt{5}}, -\frac{5}{2}\right)$; x-axis is an asymptote

(iii)



(b)



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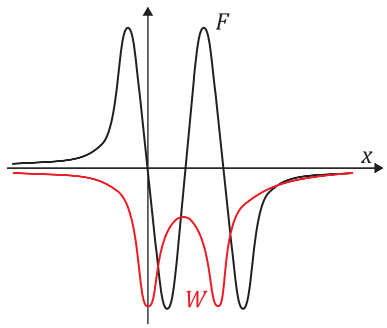

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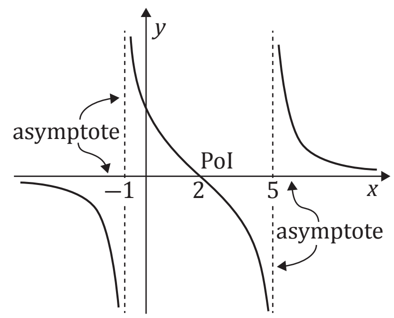
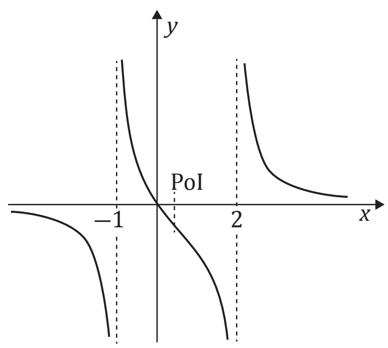


8 $\dfrac{3x}{x^2 - x - 2} \equiv \dfrac{2}{x-2} + \dfrac{1}{x+1}$

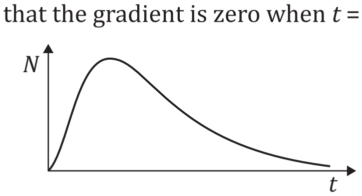
Point of inflexion at $x = \dfrac{2 - \sqrt[3]{2}}{1 + \sqrt[3]{2}} \approx \dfrac{1}{3}$



9 The graph has two vertical asymptotes, at $x = -d$ and $+d$. For $x < -d$ the potential function is as the graphs in Qs 7 and 8 (to the eye). For $-d < x < d$ the potential function is the negative of those between the asymptotes in Qs 7 and 8. It passes through (0, 0) which is also the point of inflexion. For $x > +d$ the potential function is as in Q7 and Q8 to the right of the + asymptote.

10 (a) a and b are both zero and $c = k$.
The solution with $N(0) = 0$ is $N = kt^2e^{-\lambda t}$

(b) Peak when $t = \dfrac{2}{\lambda}$
Points of inflexion when $t = \dfrac{2 \pm \sqrt{2}}{\lambda}$
Note that the gradient is zero when $t = 0$.



11 (a) $\Delta t = \dfrac{2\pi}{\omega}$

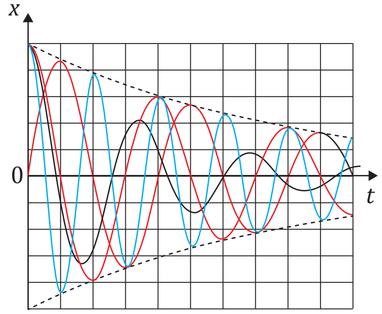
(b) Amplitude = $Ae^{-\lambda t} = Ae^{-5} \sim 0.0067\,A$

(c) $v = Ae^{-\lambda t}[\omega\cos\omega t - \lambda\sin\omega t]$

(d) $v = A\sqrt{\omega^2 + \lambda^2}\, e^{-\lambda t}\cos\left(\omega t + \tan^{-1}\frac{\lambda}{\omega}\right)$

(e) Fractional energy loss per cycle = $\left(1 - e^{-\frac{4\pi\lambda}{\omega}}\right)$

12



13 The turning points are 0.079 s earlier than those of the pure $\cos 0.2\pi t$ function.

14 (a) Minimum at (0,0); maximum at $\left(-2, \frac{4}{3}\right)$

(b) Two other points of inflexion: at $x = \frac{-3 \pm \sqrt{5}}{2}$

15 At the turning point, $v = \frac{3\sqrt{2} - 6}{2d}$.

Test Yourself 7.1

1 (a) 18.0 km N56°W (b) 70.7 N due E

(c) 55.9 N, N63.4°W (d) 44.7 N, S26.6°E

(e) 91.8 N, N15.6°E

(f) 17.3 m s^{-2}, N60°E.

(g) 100 N, N30°E.

(h) 72.1 N, E33.7°S.

2 20.6 m s^{-1} at 14.0 ° to the horizontal.

3 (a) Both components 7.07 N

(b) Down component = 453 N; up component = 211 N.

(c) N component = 2.74 km; W component = −7.52 km

4 (a) $F = 20$ N; $G = 17.3$ N (b) $F = 117$ N; $G = 110$ N

5 $F = 70$ N; $\theta = 21.8°$

6 108 N; 21.8° below the 50 N force.

7 (a) $\mathbf{F} = -2\mathbf{i} - 13\mathbf{j}$

(b) $\mathbf{F}$ = 13.2 N at 8.75° to the left of the minus $\mathbf{j}$ direction

8 (a) $\mathbf{a} = -28\mathbf{i} - 4\mathbf{j}$

(b) $\mathbf{a}$ = 28.3 m s^{-2}, W 8.1° S

9 (a) $\mathbf{s} = 20\mathbf{i} + 72\mathbf{j}$

(b) $\mathbf{v} = 10\mathbf{i} + 32\mathbf{j}$

(c) $\mathbf{v}$ = 33.5 m s^{-1} at 72.6° from the $\mathbf{i}$ vector.

10 (a) Over 0.2 s, $\bar{\mathbf{a}}$ = 120 m s^{-2}; over 0.02 s, $\bar{\mathbf{a}}$ = 124.9 m s^{-2}, both towards centre at midpoint of the time.

(b) $a = \frac{v^2}{r}$ gives $\mathbf{a}$ = 125 m s^{-2} towards centre. The mean values approach 125 as $\Delta t \to 0$.

11 T = 180 N.

12 (a) 85.4 N (b) 58.5 m

13 $F = mg\sin\theta$; $C = mg\cos\theta$ (b) $\theta_{max} = \tan^{-1} 0.2 = 11.3°$

14 a = 1.51 m s^{-2}

15 (a) $\theta = 66.9°$ (b) F = 230 N

16 (a) $40\mathbf{i} + 10\mathbf{j}$ (b) $70\mathbf{i} - 44\mathbf{j}$

(c) Both 20.6 knot (d) $14\mathbf{i} - 8.8\mathbf{j}$

(e) 16.5 knot, E 32° S

17 (a) 13 000 m s^{-1} (b) $5000\mathbf{i} + 8400\mathbf{j} + 7200\mathbf{k}$

(c) 12 140 m s^{-1}

(d) [In km] $180\,000\mathbf{i} + 367\,200\mathbf{j} + 129\,600\mathbf{k}$

(e) 429 000 km

18 (a) $\mathbf{a} = -3.7\mathbf{j}$ [m s^{-2}]; $\mathbf{v} = 30\mathbf{i} + 3\mathbf{j}$ [m s^{-1}]; $\mathbf{s} = 300\mathbf{i} + 215\mathbf{j}$ [m]

(b) 30.1 m s^{-1} at 5.7° [0.1 rad] above the horizontal

(c) 50 m s^{-1} at 53.1° below the horizontal

(d) $3\mathbf{j}$

19 (a) Position = $70.6\mathbf{i} + 70.4\mathbf{j}$, i.e. height 70.4 m and horizontal distance 70.6 m

[Position from base of cliff]

Velocity, $\mathbf{v} = 34.64\mathbf{i}$ i.e. 34.64 m s^{-1} horizontal

(b) Position = 202 m from base of cliff; $\mathbf{s} = 202\mathbf{i}$

Velocity, $\mathbf{v} = 34.64\mathbf{i} - 37.2\mathbf{j}$; i.e. 50.8 m s^{-1} at 47.1° below the horizontal.

20 (a) $\mathbf{p} = 47\mathbf{j}$

(b) $\mathbf{v}_{CoM} = 5.875\mathbf{j}$

(c) KE = 209J

21 (a) $\mathbf{p} = 24\mathbf{i} - 9\mathbf{j}$

(b) $\mathbf{v}_{CoM} = 3\mathbf{i} - 1.125\mathbf{j}$

(c) 109.5 J

22 (a) $\mathbf{p}_1 = \mathbf{p}_0 + \mathbf{F}t = 23\mathbf{i} + 25\mathbf{j}$

(b) Easiest method uses $E_k = \frac{p^2}{2m} \to \Delta E_k = 280$ J

23 (a) $\boldsymbol{u} = \frac{3}{2}\mathbf{i} + \frac{5}{2}\mathbf{j}$; $\mathbf{a} = \mathbf{i} + \mathbf{j}$

(b) $\mathbf{s} = 65\mathbf{i} + 75\mathbf{j}$

(c) $\mathbf{F.s} = (2\mathbf{i} + 2\mathbf{j}).(65\mathbf{i} + 75\mathbf{j}) = 130 + 150 = 280$ J

Comment: **F.s** is the work done by the force which is the change in kinetic energy, i.e. the answer agrees with Q22 (b)

24 $\mathbf{F}.\Delta\mathbf{s} = 600$ J $\therefore$ Final KE = 1 000 J

25 (a) $\tau_1 = 30\mathbf{k}$; $\tau_2 = -16\mathbf{k}$

(b) $-14\mathbf{k}$

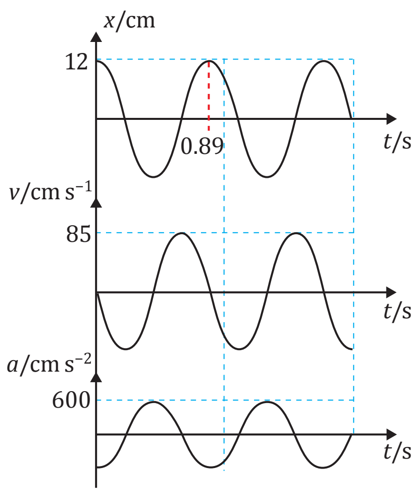
(c) $\mathbf{F}_3 = -4\mathbf{i} - 4\mathbf{j}$

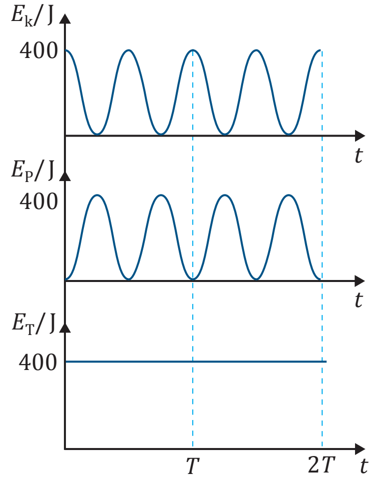
(d) $(x\mathbf{i} + y\mathbf{j}) \times \mathbf{F}_3 = (x\mathbf{i} + y\mathbf{j}) \times (-4\mathbf{i} - 4\mathbf{j}) = (-4x + 4y)\mathbf{k}$

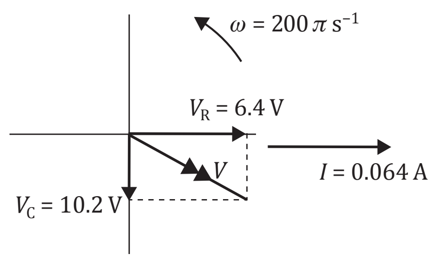
This cross product must be $-14\mathbf{k}$

$\therefore -4x + 4y = -14$, i.e. $x - y = 3.5$

Test Yourself 8.1

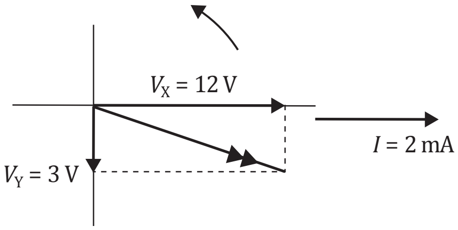
1. (a) 0.909 (b) −23.4 (c) 15.0
2. (a) −5.488 rad; −0.795 rad; 0.795 rad; 5.488 rad
   (b) −2.214 rad; −0.927 rad; 4.069 rad; 5.356 rad
   (c) −3.094 rad; 1.094 rad
3. 6.79×10^{-5} rad
4. (a) 1 pc = 3.08×10^{13} km
   (b) 1 pc = 3.25 l-y
5. 0.015%
6. (a) $x = 10\cos 2\pi t$
   (b) $x = -10\cos 2\pi t$ or $x = 10\cos(2\pi t \pm \pi)$
   (c) $x = 10\sin 2\pi t$ or $x = 10\cos\left(2\pi t - \frac{\pi}{2}\right)$
   (d) $x = 10\sin(2\pi t - 1.8\pi)$ or $x = 10\cos(2\pi t - 0.3\pi)$
   NB There are other ways of expressing these functions
7. (a) $v_{max} = 20\pi = 62.8$ cm s^{-1}; $a_{max} = 40\pi^2 = 396$ cm s^{-2}
   (b) for 6(a): v_{max} at −0.25 s and 0.75 s; a_{max} at −0.5 s and 0.5 s
   for 6(b): v_{max} at 0.25 s and −0.75 s; a_{max} at −1 s, 0 and 1 s
   for 6(c): v_{max} at −1 s, 0 and 1 s; a_{max} at −0.25 s and 0.75 s
   for 6(d): v_{max} at −0.1 s and 0.9 s; a_{max} at −0.35 s and 0.65 s
8. $\omega = \sqrt{\frac{k}{m}} = \sqrt{50} = 7.07$ s^{-1}; $f = \frac{\omega}{2\pi} = 1.125$ Hz; $T = \frac{1}{f} = 0.889$ s;
   $A = 12$ cm
9. 

10. $x = 12\cos 7.07t$; $v = -85\sin 7.07t$; $a = -600\cos 7.07t$
    [in cm; again there are several ways of writing these, e.g. $v = 85\cos\left(7.071t + \frac{\pi}{2}\right)$]
11. $x = 2.82$ cm; $v = 82.5$ cm s^{-1}; $a = -141$ cm s^{-2}.
12. 0.556 s, 0.778 s, 1.444 s and 1.667 s
13. K.E $= \frac{1}{2}mv^2 = \frac{1}{2} \times 0.5 \times (0.849\sin 7.071t)^2 = 0.176$ J
    P.E. $= \frac{1}{2}kx^2$: Extension $= \frac{mg}{k} - 0.0187 = 0.178$ m
    ∴ PE = 0.396 J
14. (a) Max velocity $= A\omega = 2.0 \times 10 = 20$ m s^{-1}.
    This occurs at $t = 0$.
    ∴ K.E (0) $= \frac{1}{2} \times 2 \times 20^2 = 400$ J. This is the maximum K.E.
    (b) 

15. (a) −0.192 s; −0.058 s; 0.008 s; 0.142 s.
    (b) −0.196 s; −0.154 s; 0.004 s; 0.046 s
16. (a) $I = 0.12\cos 200\pi t$
    (b) (i) $V = 3.71$ V, (ii) $I = 0.037$ A, (iii) $P = 0.138$ W
    (c) (i) $V_{rms} = 8.49$ V, (ii) $I_{rms} = 0.0849$ A, (iii) $\langle P \rangle = 0.720$ W.
17. (a) (i) $X_C = \frac{1}{\omega C} = 159\ \Omega$, (ii) $I_0 = \frac{V_0}{X_C} = 0.075$ A
    (b) $I = 0.075\cos\left(200\pi t + \frac{\pi}{2}\right)$
    (c) $I = -0.071$ A
18. (a) $I = 0.191\cos\left(200\pi t - \frac{\pi}{2}\right)$
    (b) $I = 0.182$ A
19. (a) (i) $Z = \sqrt{100^2 + 159^2} = 188\ \Omega$,
    (ii) $I_0 = 0.064$ A,
    (iii) $V_R = 6.4$ V; $V_C = 10.2$ V
    (b) $V = \sqrt{10.2^2 + 6.4^2} = 12$ V
    

    (c) $\theta = \tan^{-1}\left(\frac{10.2}{6.4}\right) = 1.01$ rad
20. (a) X is a resistor because V is in phase with I; Y is a capacitor because I leads V by 90°.
    (b) $R = \frac{V_R}{I} = \frac{12}{2 \times 10^{-3}} = 6$ kΩ; $\frac{1}{\omega C} = \frac{V_C}{I}$
    $\therefore C = \frac{I}{\omega V_C} = \frac{2 \times 10^{-3}}{500 \times 6} = 0.67\ \mu$F/670 nF
    (c) Applied voltage $= \sqrt{12^2 + 6^2} = 13.4$ V.;
    angle = 0.464 rad (= 26.6°)

21 (a) V_X is unchanged at 12 V because resistance is constant. V_Y is halved to 3 V because capacitor reactance is inversely proportional to frequency.



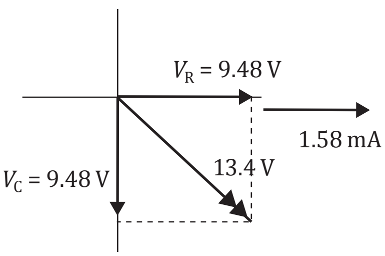
(b) $V = 12.4$ V; $\phi = 0.245$ rad (14.0°)

22 Method: $X_C = \dfrac{1}{250 \times 0.67 \times 10^{-6}} = 6000\ \Omega$

$Z = \sqrt{R^2 + X^2} = \sqrt{6^2 + 6^2} = 8.49\ \text{k}\Omega$

$\therefore\ I = 1.58$ mA

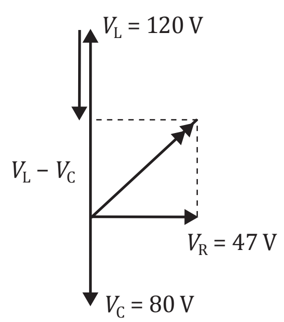
$\therefore\ V_R = 9.48$ V; $V_C = 9.48$ V; $V = 13.4$ V



23 (a) Method: $V_R = IR = 0.1 \times 470 = 47$ V;

$V_C = \dfrac{1}{\omega C} = \dfrac{0.1}{500 \times 2.5 \times 10^{-6}} = 80$ V

$V_L = I\omega L = 0.1 \times 500 \times 2.4 = 120$ V.



(b) $V = \sqrt{47^2 + (120 - 80)^2} = 61.7$ V

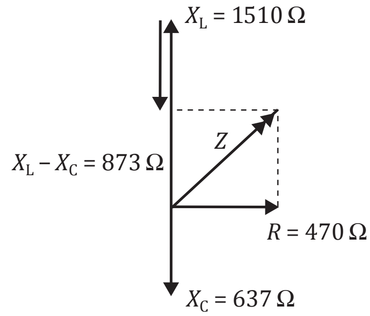
(c) $\langle P \rangle = I^2R$ [rms current] $= 0.1^2 \times 470 = 4.7$ W

[or $V_R I = 47 \times 0.1 = 4.7$ W]

24 Method: $X_L = \omega L = 2\pi \times 100 \times 2.4 = 1510\ \Omega$

$X_C = \dfrac{1}{\omega C} = \dfrac{1}{2\pi \times 100 \times 2.5 \times 10^{-6}} = 637\ \Omega.$

$Z = \sqrt{R^2 + (X_L - X_C)^2} = \sqrt{470^2 + 873^2} = 991\ \Omega$



$\therefore\ I = \dfrac{V}{Z} = \dfrac{40}{991} = 0.040$ A

25 (a) $\omega = \dfrac{1}{\sqrt{LC}} = \dfrac{1}{\sqrt{2.4 \times 2.5 \times 10^{-6}}} = 408\ \text{s}^{-1}.\ \therefore\ f = \dfrac{\omega}{2\pi} = 65.0$ Hz

(b) The reactances of the inductor and capacitor are equal and opposite, so $Z = R$.

$\therefore\ I = \dfrac{V}{R} = \dfrac{50}{470} = 0.106$ A.

(c) $V_R = 50$ V; $V_L = I\omega L = 0.106 \times 408 \times 2.4 = 104$ V;

$V_C = V_L = 104$ V

[Alternatively calculate V_C using $V_C = \dfrac{I}{\omega C}$]

(d) Only the resistor dissipates power,

so $\langle P \rangle = I_{rms}{}^2 R = 0.106^2 \times 470 = 5.3$ W

Test Yourself 9.1

1 **E** = 3 kV m^{-1} downwards [or 3 kN C^{-1}]

2 **E** = 980 V m^{-1} upwards

3 9.0×10^{24} kg

4 40 000 km from the Moon on the line joining the centres of the Earth and Moon.

5 $V_G = -1.13 \times 10^6$ J kg^{-1}

6 Acceleration due to Sun = 2.4 × acceleration due to Earth

[NB This means that the Moon's path is always concave to the Sun]

7 1.0×10^6 m s^{-1} at 10.0° to original direction [0.174 rad]

8 4.5 MV m^{-1}

9 450 kV

10 22 pF

11 If the sphere carries a charge, Q, the potential, $V = \dfrac{1}{4\pi\varepsilon_0}\dfrac{Q}{a}$.

$\therefore\ \dfrac{Q}{V} = C = 4\pi e_0 a$

12 Field is radial, so at right angles to the curved surface of an imaginary concentric cylinder.

$\therefore$ Flux emerging from cylinder $= E2\pi rl = \dfrac{Q}{\varepsilon_0}$

$Q = 3 \times 10^{-6}\,l.\ \therefore\ E2\pi \times 0.1l = \dfrac{3 \times 10^{-6}l}{8.854 \times 10^{-12}}$,

leading to $E = 540$ kV m^{-1}.

13 2.4 μC m^{-2}

14 Method: Use vector equilibrium to find the horizontal force on each sphere [0.253 mN]

Then use $F = \frac{1}{4\pi\varepsilon_0}\frac{QQ}{r^2} \rightarrow Q = 16.8\text{ nC}$

15 (a) E due to each = 60 500 V m^{-1} in opposite directions.

(b) Resultant field = 0

16 $V = 3024 + 3024 = 6050\text{ V}$

17 $W = \frac{1}{4\pi\varepsilon_0}\frac{Q_1Q_2}{d^2} = 9 \times 10^9 \times \frac{(16.8 \times 10^{-9})^2}{0.1}$

$= 2.54 \times 10^{-5}\text{ J} \sim 25\ \mu\text{J}$

18 Electrical potential energy 10 cm apart = 2.54×10^{-5} J

∴ Electrical PE 5 cm apart = 5.08×10^{-5} J

∴ Loss in electrical PE = 2.54×10^{-5} J

Gain in height between the two positions = 0.48 cm [needs calculating]

∴ Gain in gravitational potential energy = $mg\Delta h = 9.4 \times 10^{-6}$ J

∴ Loss in PE = $2.54 \times 10^{-5} - 9.4 \times 10^{-6}$ J = 1.60×10^{-5} J

Using KE = $\frac{1}{2}mv^2$ we get $v = 0.4$ m s^{-1}.

19 (a) 121 000 V m^{-1} (b) 0

20 (a) 5.95×10^{24} kg (b) 9.78 N kg^{-1} [Both correct to 2 s.f.]

21 Total mass within outer core boundary = 1.98×10^{24} kg

This gives g = 10.8 N kg^{-1}

The uniform density value would be $\frac{3500}{6370} \times 9.8 = 5.4$ N kg^{-1}, i.e. true value ~ 2× uniform density value.

22 (a) acceleration = 8.8×10^{12} m s^{-2}

(b) radius of circle = 1.14 mm

(c) 14 MHz

23 The force due to the magnetic field provides the centripetal force.

$\therefore \frac{mv^2}{r} = Bqv.\ \therefore \omega = \frac{v}{r} = \frac{Bq}{v}.$

$\therefore f = \frac{Bq}{2\pi m}$, which is independent of the speed.

For a proton with $m = 1.67 \times 10^{-27}$ kg, f = 460 Hz.

24 Peak current = $I_0 \times \sqrt{2} = 28.3$ A.

$F_{max} = BI\ell\cos\theta = 5 \times 10^{-5} \times 28.3 \cos 60° = 0.7$ mN; f = 50 Hz

∴ ω = 100π

∴ F / mN = $0.7\cos(100\pi t + \varepsilon)$

25 Algebraically, induced emf $\mathcal{E}_{in} = \frac{\Delta(N\Phi)}{t} = B\ell v$

$\therefore I = \frac{B\ell v}{R}$. So the motor force, $BI\ell = \frac{B^2\ell^2 v}{R}$.

∴ The work done per second, $P = \frac{B^2\ell^2v^2}{R}$

The electrical power $P = I^2R = \left(\frac{B\ell v}{R}\right)^2 R = \frac{B^2\ell^2v^2}{R}$, which is the same.

Numerically in Example G, both powers are 0.05 W.

Data Exercise 10.1

$x_1 = 2.0$; $y_1 = 14.0$

x_2	y_2	Δx	Δy	$\frac{\Delta y}{\Delta x}$
3.0	29.00	1.0	15	15
2.5	20.75	0.5	6.75	13.5
2.1	15.23	0.1	1.23	12.3
2.05	14.6075	0.05	0.6075	12.15
2.01	14.1203	0.01	0.1203	12.03
2.005	14.06008	0.005	0.060075	12.015
2.001	14.012	0.001	0.012003	12.003

As $\Delta x \rightarrow 0$, $\frac{\Delta y}{\Delta x}$ appears to tend to 12. This is confirmed by the fact that if $\Delta x = -0.001$, $\frac{\Delta y}{\Delta x} = 11.997$.

Data Exercise 10.3

No. of strips	Lower area (A_L)	Upper area (A_U)
10	6.84	9.24
20	7.41	8.61
100	7.88	8.12
200	7.94	8.06
1000	7.988	8.012

Test Yourself 10.1

1 $\frac{dy}{dx} = 75x^2 = 168.75$

2 $\frac{dx}{dt} = 15\cos t = -15$

3 $\frac{dN}{dt} = 600e^t = 989$

4 $\frac{dy}{dx} = \frac{6.0}{t} = 1.0$

5 $\frac{dy}{dx} = 8 + 6t = 23$

6 $\frac{dx}{dt} = -3\sin t + 8\cos t = -3.00$

7 $\frac{dy}{dx} = 6x^2 + 3e^x = 26.9$

8 $\frac{dx}{dt} = \frac{10}{t} - \frac{1.5}{\sqrt{t}} = 1.75$

9 $\frac{dy}{dx} = 10x^2(x^2 - 3)$

10 $\frac{dy}{dx} = 3x^2 + 6x - 5$

11 $\frac{dy}{dx} = x(2 + x)e^x$

12 $\frac{dy}{dx} = 6t\sin t + 3t^2\cos t$

13 $\frac{dy}{dx} = \frac{-2}{(x-1)^2}$

14 $\frac{dy}{dx} = \frac{x^2 - 4x - 7}{(x-2)^2}$

15 $\frac{dx}{dt} = \frac{2t^3(4\cos t + t\sin t)}{\cos^2 t}$

16 $\frac{dy}{dx} = \frac{1 - 2\ln x}{x^3}$

17 (b) $\frac{df}{dg} = 3g^2 = 3(x^2 + 2)^2$; $\frac{dg}{dx} = 2x$

(c) $\frac{dy}{dx} = 3(x^2 + 2)^2 2x = 6x^5 + 24x^3 + 24x$

Check: $f = (x^2 + 2)^3 = x^6 + 6x^4 + 12x^2 + 8$

$\therefore \frac{df}{dx} = 6x^5 + 24x^3 + 24x$ QED.

18 $\dot{x} = 75\cos 3t$

19 $\dot{v} = -31\,400\sin\left(314t - \frac{\pi}{4}\right)$

20 $\frac{dN}{dt} = -1.0 \times 10^{11} e^{-0.1t}$

21 $\frac{dQ}{dt} = -\frac{1}{5}e^{-t/25}$

22 $\frac{dV}{dR} = \frac{Er}{(R+r)^2}$. When $R = 0$, gradient $= \frac{dV}{dR} = \frac{E}{r}$

23 (a) $\frac{dQ}{dt} = -\frac{Q_0}{RC}e^{-t/RC}$

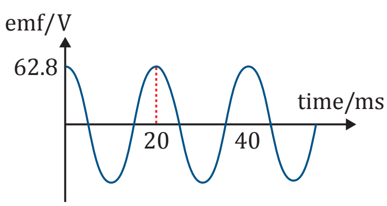
(b) $\frac{dQ}{dt} = -0.0092$

(c) (−)9.2 mA

24 $E_{IN} = BAN\omega\cos\omega t = 62.8\cos 100\pi t$.

∴ Peak voltage = 62.8 V

Period: $100\pi T = 2\pi$ ∴ $T = 0.02$ s = 20 ms

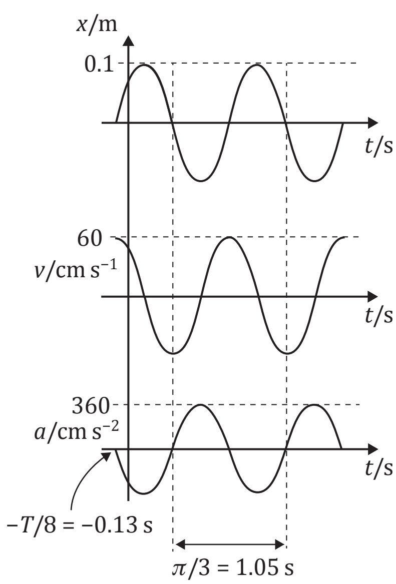


25 (a) $v = A\omega\cos\left(\omega t + \frac{\pi}{4}\right)$ $\left[= 60\cos\left(6t + \frac{\pi}{4}\right)\right]$

(b) $a = -A\omega^2\sin\left(\omega t + \frac{\pi}{4}\right)$ $\left[= -360\sin\left(6t + \frac{\pi}{4}\right)\right]$

(c) See graphs

(d) 0.36 N



Test Yourself 10.2

1 $f(x) = \frac{25}{6}x^6$

2 $f(x) = -\frac{6}{x^2} + 8$

3 $f(t) = 2t^2 + t + 4$

4 $f(t) = -500e^{-0.005t}$

5 $f(t) = -2\cos 2.5t + 8\sin 1.25t$

6 $\left[2x^3\right]_2^4 = 128 - 16 = 112$

7 $\left[-2.5e^{-2t}\right]_0^{0.5} = 1.58$

8 $\left[10\ln x\right]_1^5 = 16.1$

9 $\left[\frac{2}{\pi}\sin\pi t + 2x\right]_0^2 = 4$

10 $\left[-x^{-3}\right]_1^{\infty} = 1$

11 $[k = 5000]$ $W = 5493$ J

12 $[k = 792.4]$ $W = 4445$ J

13 $[k = 2.048 \times 10^{17}\text{ N m}^2]$. $W = \int_{r_1}^{r_2}\frac{k}{r^2}dr = \left[-\frac{k}{r}\right]_{r_1}^{r_2} = k\left(\frac{1}{r_1} - \frac{1}{r_2}\right)$

$= 1.5 \times 10^{10}$ J

14 $W = \int_{r_1}^{r_2}\frac{GMm}{x^2}dx = \left[-\frac{GMm}{x}\right]_{r_1}^{r_2} = GMm\left(\frac{1}{r_1} - \frac{1}{r_2}\right)$

15 $W = -\frac{GMm}{a}$

16 $V_G = -\frac{GM}{a}$

17 $V_E = \frac{1}{q}\int_{\infty}^{a} -\frac{Qq}{4\pi\varepsilon_0 x^2}dx = \frac{Q}{4\pi\varepsilon_0}\left[\frac{1}{x}\right]_{\infty}^{a} = \frac{Q}{4\pi\varepsilon_0 a}$

18 Potential energy at a $\int_{\infty}^{a}\frac{Qd}{2\pi\varepsilon_0 x^3}dx = \left[-\frac{Qd}{4\pi\varepsilon_0 x^2}\right]_{\infty}^{a} = -\frac{Qd}{4\pi\varepsilon_0 a^2}$

∴ KE at $a = \frac{Qd}{4\pi\varepsilon_0 a^2}$ ∴ $v = \sqrt{\frac{Qd}{2\pi\varepsilon_0 m a^2}}$

19 (a) $N_0 = \int_0^{\infty} Ae^{-\lambda t} = \frac{A_0}{\lambda}$ (b) $N_0 = 1.75 \times 10^{16}$.

20 (a) $\frac{dQ}{dt} = -I_0 e^{-t/RC}$, ∴ $Q = \int\left(-I_0 e^{-t/RC}\right)dt = I_0 RCe^{-t/RC} + c$

Putting $Q = 0$ when $t = \infty \rightarrow c = 0$, i.e. $Q = I_0 RCe^{-t/RC}$

(b) $Q_0 = I_0 RC$ (c) $Q(200) = 68$ mC.

21 (a) Substituting $(0, h)$ into $F = a - bt^2 \rightarrow h = a$.

Substituting $(\lambda, 0)$ into $F = a - bt^2$

$\rightarrow 0 = a - b\lambda^2 = h - b\lambda^2$ (from above)

∴ $b = \frac{h}{\lambda^2}$ ∴ $F = h - \frac{h}{\lambda^2}t^2 = h\left(1 - \frac{t^2}{\lambda^2}\right)$ QED

(b) $\Delta p = \frac{4\lambda h}{3}$ (c) $h = 5740$ N

22 (a) $x = \int 25e^{-0.02t}dt = -1250e^{-0.02t} + c$; Applying initial conditions gives $x = 1250(1 - e^{-0.02t})$.

(b) $D = x$ at $t = \infty$; ∴ $D = 1250$ m

(c) $v = 25 - \frac{x}{50}$

23 (a) $\Delta M = \frac{\Delta x}{l}M$

(b) $\Delta E_k = \frac{1}{2}\frac{M\omega^2}{l}x^2\Delta x$

(c) $E_k = \int_0^l \frac{1}{2}\frac{M\omega^2}{l}x^2\,dx = \frac{1}{6}Ml^2\omega^2$

(d) $I = \frac{1}{3}Ml^2$

24 Dividing by the $\frac{1}{2}\omega^2$ term from the start:

$$I = \int_{-l/2}^{l/2} \frac{M}{l} x^2\,\mathrm{d}x = \tfrac{1}{12} M l^2$$

25 (a) Area of ring = $2\pi r \Delta r$.

$\therefore$ Mass of ring, $\Delta M = \dfrac{2\pi r \Delta r}{\pi a^2} M = \dfrac{2r\Delta r}{a^2} M$

(b) $\Delta E_\mathrm{k} = \dfrac{M r^3 \omega^2 \Delta r}{a^2}$, so $E_\mathrm{k} = \dfrac{M\omega^2}{a^2}\displaystyle\int_0^a r^3\mathrm{d}r = \tfrac{1}{4} M a^2 \omega^2$

(c) $I = \tfrac{1}{2} M a^2$

Test Yourself 11.1

1 $v = 10e^{-5t}$

2 $N = 1 \times 10^6 e^{-0.001t}$

3 I / μA = $6e^{-0.2t}$

4 $x = 5 \sin 8t$

5 $x = 0.1 \cos 5t$

6 $h = 50e^{-0.02t}$

7 $V = 9e^{-0.097t}$

8 $y = 0.224 \cos(10t - 1.11)$ or $y = 0.224 \sin(10t + 0.46)$

9 Q / μC = $0.2 \sin 500t$

10 Q / mC = $47 \cos 100t$

$\therefore$ T = 0.0628 s

11 $v = 50 - 30e^{-0.1t}$

12 $N = \dfrac{R}{\lambda}(1 - e^{-\lambda t})$

13 $V = 16e^{-0.3t} + 24$

14 $I = 0.5 \sin 13t + 1.2 \sin 12t$

15 $x = 0.2(1 - \cos 10t)$

16 $v = 12.5(1 - e^{-0.4t}) + 5t$

17 $v = 0.206e^{-12t} + 0.443 \sin(2\pi t - 0.482)$

18 $N = 2 \times 10^7(e^{-0.2t} - e^{-0.1t})$

19 $N_\mathrm{B} = 1 \times 10^{15}(e^{-0.05t} - \mathrm{e}^{-0.125t})$ with t in days.

$\therefore N_\mathrm{B}$ (20 days) = 2.9×10^{14}

20 x / m = $0.1 \cos 2.5t$

21 $x = 2.0e^{-2t} \cos 9.80t$

22 (a) $\dfrac{\mathrm{d}^2x}{\mathrm{d}t^2} + 4\dfrac{\mathrm{d}x}{\mathrm{d}t} + 9x = 0$ (b) $x = -0.15e^{-2t}\cos\sqrt{5}t$

23 (a) $k = 0.1$; $p = \pi$, i.e. 3.1420; $\omega = 3.1420$

$$\frac{\mathrm{d}^2x}{\mathrm{d}t^2} + 0.1\frac{\mathrm{d}x}{\mathrm{d}t} + 9.872x = 0$$

(b) $x = 0.615e^{-0.05t} \sin \pi t$

(c) 0.00026 s

[without damping the period would be 1.99974 s]

24 (a) $L\dfrac{\mathrm{d}I}{\mathrm{d}t} + IR = V_0 \cos\omega t$ or $\dfrac{\mathrm{d}I}{\mathrm{d}t} + I\dfrac{R}{L} = \dfrac{V_0}{L}\cos\omega t$

(b) Phase difference, $\varepsilon = \tan^{-1}\left(\dfrac{\omega L}{R}\right)$

(c) $V_0 = I_0\sqrt{\omega^2 L^2 + R^2}$

25 (a) If $L\dfrac{\mathrm{d}I}{\mathrm{d}t} + IR + \dfrac{Q}{C} = V_0 \cos\psi t$,

differentiating $\rightarrow L\dfrac{\mathrm{d}^2I}{\mathrm{d}t^2} + R\dfrac{\mathrm{d}I}{\mathrm{d}t} + \dfrac{I}{C} = -V_0\psi \sin\psi t$

Look for a CF of the form $I = I_0 \cos(\psi t + \varepsilon)$

Substituting into the differential equation gives:

$$-V_0\psi \sin\psi t = -I_0 L\psi^2 \cos(\psi t + \varepsilon) - I_0 R\psi \sin(\psi t + \varepsilon) + \frac{I_0}{C}\cos(\psi t + \varepsilon)$$

$$\therefore -V_0 \sin\psi t = I_0\left[\frac{1}{\psi C} - \psi L\right]\cos(\psi t + \varepsilon) - I_0 R \sin(\psi t + \varepsilon)$$

To find the values of I_0 and ε, consider substitute the following values of t:

$t = 0$: $\quad 0 = \left[\dfrac{1}{\psi C} - \psi L\right]\cos\varepsilon - R\sin\varepsilon$ [dividing by I_0]

$$\therefore \tan\varepsilon = \frac{\frac{1}{\psi C} - \psi L}{R}$$

$$\therefore \sin\varepsilon = \frac{\frac{1}{\psi C} - \psi L}{\sqrt{R^2 + \left(\frac{1}{\psi C} - \psi L\right)^2}} \text{ and } \cos\varepsilon = \frac{R}{\sqrt{R^2 + \left(\frac{1}{\psi C} - \psi L\right)^2}}$$

$\psi t = \frac{\pi}{2}$ $\quad -V_0 = -I_0\left[\dfrac{1}{\psi C} - \psi L\right]\sin\varepsilon - I_0 R\cos\varepsilon$

$$\therefore V_0 = I_0\left\{\left[\frac{1}{\psi C} - \psi L\right]\frac{\frac{1}{\psi C} - \psi L}{\sqrt{R^2 + \left(\frac{1}{\psi C} - \psi L\right)^2}} + R\frac{R}{\sqrt{R^2 + \left(\frac{1}{\psi C} - \psi L\right)^2}}\right\}$$

$$\therefore = I_0\left\{\frac{R^2 + \left(\frac{1}{\psi C} - \psi L\right)^2}{\sqrt{R^2 + \left(\frac{1}{\psi C} - \psi L\right)^2}}\right\} = I_0\sqrt{R^2 + \left(\frac{1}{\psi C} - \psi L\right)^2} \quad \text{QED}$$

(b) $V_0 = I_0\sqrt{R^2 + \left(\psi L - \dfrac{1}{\psi C}\right)^2}$.

$\therefore I_0$ is maximum when $\psi L - \dfrac{1}{\psi C} = 0$

$\therefore \psi^2 = \dfrac{1}{LC}$ $\quad \therefore \psi = \dfrac{1}{\sqrt{LC}}$ $\quad \therefore f_\mathrm{R} = \dfrac{1}{2\pi\sqrt{LC}}$

Test Yourself 12.1

1 (a) 3i; −3i (b) −4 + 5i

(c) (i) 1 (ii) i (iii) −1 (iv) −i

(d) (i) 1 (ii) −1 (iii) −i (iv) i

(e) (i) 0 (ii) 1 (iii) −1

2 (a) Re (z^*) = 4 (b) Im (z^*) = 3 (c) Re (iz) = −3

(d) Im (iz) = 4

3 (a) x = 3 + 2i and 3 − 2i

(b) −3 + 2i and −3 − 2i respectively

(c) $x^2 + 6x + 13 = 0$

4 2i

5 (a) 13i (b) i (c) $-\frac{1}{2}+\frac{\sqrt{3}}{2}\text{i}$
(d) $(a^2-b^2)+2ab\text{i}$

6 (a) 5 (b) $\frac{4}{25}-\frac{3}{25}\text{i}$ (c) $\frac{1}{25}\sqrt{3^2+4^2}=\frac{1}{5}$
(d) $\frac{1}{5}$

7 (a) i (b) $\frac{1}{2}\left(\sqrt{3}+1\right)+\frac{1}{2}\left(\sqrt{3}-1\right)\text{i}$

8 (a) (i) $\sqrt{3}+\text{i}$ (ii) $2+2\sqrt{3}\text{i}$ (iii) $-2+2\sqrt{3}\text{i}$
(iv) $-4-4\sqrt{3}\text{i}$
(b) (i) $\sqrt{2}e^{\text{i}\frac{\pi}{4}}$ (ii) $\sqrt{2}e^{\text{i}\frac{3\pi}{4}}$ (iii) $\sqrt{2}e^{-\text{i}\frac{3\pi}{4}}$
(iv) $\sqrt{2}e^{-\text{i}\frac{\pi}{4}}$
(v) $2e^{-\text{i}\frac{\pi}{6}}$ (vi) $2e^{-\text{i}\frac{5\pi}{6}}$

9 (a) $8e^{\text{i}\frac{\pi}{2}}=8\text{i}$ (b) $\frac{1}{2}e^{-\text{i}\frac{\pi}{6}}$
(c) $\sqrt{2}e^{\text{i}\frac{\pi}{4}}\times 2e^{-\text{i}\frac{\pi}{6}}=2\sqrt{2}e^{\text{i}\frac{\pi}{12}}$ (d) $\frac{\sqrt{2}e^{\text{i}\frac{\pi}{4}}}{\sqrt{2}e^{-\text{i}\frac{\pi}{4}}}\times=e^{\text{i}\frac{\pi}{2}}=\text{i}$

10 $\sin(A+B)+\sin(A-B)$

11 $\cos 5\theta=16\cos^5\theta-20\cos^3\theta+5\cos\theta$

12 (a) $\sqrt{-8}=-2\text{i},\ 2e^{\text{i}\frac{\pi}{3}},\ 2e^{-\text{i}\frac{\pi}{3}}$ (b) $2,\ 1+\sqrt{3}\text{i},\ 1-\sqrt{3}\text{i}$

Test Yourself 12.2

1 (a) $z_1=Ae^{\text{i}\left(\omega t+\frac{\pi}{2}\right)}=Ae^{\text{i}\omega t}e^{\text{i}\frac{\pi}{2}}=A\text{i}e^{\text{i}\omega t};$
$z_2=3Ae^{\text{i}(\omega t+\pi)}=3Ae^{\text{i}\omega t}e^{\text{i}\pi}=-3Ae^{\text{i}\omega t}$
(b) $(z_1+z_2)=Ae^{\text{i}\omega t}(-3+\text{i})=Ae^{\text{i}\omega t}\sqrt{10}e^{\text{i}\phi}$
where $\phi=\cos^{-1}\left(-\frac{3}{\sqrt{10}}\right)=0.898\pi$ [= 2.820 rad]
(c) $(x_1+x_2)=\sqrt{10}A\cos\{\omega t+0.898\pi\}$

2 (a) Using Newton's 2nd law in basic SI units the differential equation is:
$$1.50\frac{\text{d}^2x}{\text{d}t^2}=-3.6\frac{\text{d}x}{\text{d}t}-96x$$
which reduces to $\frac{\text{d}^2x}{\text{d}t^2}+k\frac{\text{d}x}{\text{d}t}+\omega_0{}^2=0,$

where $\omega_0=\sqrt{\frac{96}{1.5}}=\sqrt{64}=8.0\text{ s}^{-1}$ and $k=\frac{3.6}{1.5}=2.4\text{ s}^{-1}$

(b) $\lambda^2+2.4\lambda+64=0$
$$\therefore\lambda=\frac{-2.4\pm\sqrt{2.4^2-4\times 64}}{2}=-1.2\pm 7.91\text{i}$$
$\therefore$ Re $(\lambda)=-1.2$; Im $(\lambda)=\pm 7.91$

(c) $\omega_1=7.91,\ \therefore T=\frac{2\pi}{\omega_1}=0.79$ s (2 s.f.)

(d) If $e^{-1.2t}=0.05;\ t=\frac{\ln 0.05}{-1.2}=2.5$ s (2 s.f.) $=3.1T$.

So 3 complete cycles.

3 (a) (i) $\boldsymbol{Z}_\text{S}=R-\text{i}X_\text{C}=R-\frac{\text{i}}{\omega C}$
(ii) $\boldsymbol{Z}_\text{P}=\frac{R(\text{i}X_\text{C})}{R-\text{i}X_\text{C}}=\frac{R}{1+(\omega CR)^2}[1-\text{i}\omega CR]$
(b) (i) $\boldsymbol{Z}_\text{S}=R-\text{i}R=R\sqrt{2}e^{-\text{i}\frac{\pi}{4}}$
(ii) $\boldsymbol{Z}_\text{P}=\frac{R}{2}[1-\text{i}]=\frac{R}{\sqrt{2}}e^{-\text{i}\frac{\pi}{4}}$
(c) $\frac{I_\text{S}}{I_\text{P}}=\frac{Z_\text{P}}{Z_\text{S}}=\frac{1}{2}$; Phases the same $\left[\text{leading the pd by }\frac{\pi}{4}\right]$

4 (a) (i) $\boldsymbol{Z}=R+\text{i}\left(\omega L-\frac{1}{\omega C}\right)$
(ii) $Z=\sqrt{R^2+\left(\omega L-\frac{1}{\omega C}\right)^2}$
(b) The minimum value of Z is when $\left(\omega L-\frac{1}{\omega C}\right)=0,$
$\therefore\omega_0L=\frac{1}{\omega_0C},\ \therefore\omega_0=\frac{1}{\sqrt{LC}}$
At this frequency $Z=\sqrt{R^2+0}=\sqrt{R^2}=R$
(c) $\frac{\text{peak pd across }L}{\text{peak pd across }R}=\frac{I\omega_0L}{IR}=\frac{\omega_0L}{R}$
(d) If $\sqrt{2}R=\sqrt{R^2+\left(\omega L-\frac{1}{\omega C}\right)^2}$, then $2R^2=R^2+\left(\omega L-\frac{1}{\omega C}\right)^2$
$\therefore\left(\omega L-\frac{1}{\omega C}\right)^2=R^2$, so $\omega L-\frac{1}{\omega C}=\pm R$ QED
(e) $\omega_1L-\frac{1}{\omega_1C}=-R$ and $\omega_2L-\frac{1}{\omega_2C}=+R$
(i) $\therefore$ adding: $(\omega_1+\omega_2)L-\frac{\omega_1+\omega_2}{\omega_1\omega_2C}=0,$
i.e. $(\omega_1+\omega_2)\left[L-\frac{1}{\omega_1\omega_2C}\right]=0$
$\therefore L-\frac{1}{\omega_1\omega_2C}=0$ leading to $\omega_1\omega_2=\frac{1}{LC}=\omega_0{}^2$
(ii) and subtracting $(\omega_2-\omega_1)L+\frac{\omega_2-\omega_1}{\omega_1\omega_2C}=2R.$
but $\omega_1\omega_2=\frac{1}{LC}$ so $2L(\omega_2-\omega_1)=2R$
So $\frac{\omega_2-\omega_1}{\omega_0}=\frac{R}{\omega_0L}=\frac{1}{Q}$
(f) From (e)(ii) Q is inversely proportional to the fractional difference of the half power points related to the resonant frequency. So the sharper the resonance peak, the greater the value of Q.